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Scattering of gravity waves by a circular dock

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The scattering of a gravity wave of wave number k by a circular dock of radius a and draft $d-h$ in water of depth d is calculated through a variational approximation. The total and differential scattering cross-sections, the peripheral displacement, and the lateral force on the dock are presented as functions of ka with d/a and h/d as parameters and compared with the classical results for a circular cylinder ($h = 0$). A pronounced resonance is found near $ka = 2$ for certain values of d/a and h/d.

1. Statement of problem

We consider the scattering of a gravity wave of amplitude ζ_0 and period $2\pi/\sigma$ by a circular dock of radius a and draft $d-h$ in water of depth d (see figure 1; we disregard any supporting structure in $z < h$)§. We assume small displacements and irrotational flow, so that the motion of the water is governed by the classical linearized theory (Lamb 1932, chapter 9). The problem for $h = 0$ is equivalent to the acoustical problem of the scattering of a plane wave by a circular cylinder (Lamb, §304).

Following the usual convention for simple harmonic motions, we suppose that the free-surface displacement is given by the real part of $\zeta \exp(-i\sigma t)$ and pose the incident wave in the alternative forms

$$\zeta_i = \zeta_0 e^{ikx} \tag{1.1a}$$

$$= \zeta_0 \sum_{m=0}^{\infty} \epsilon_m i^m J_m(kr) \cos m\theta \tag{1.1b}$$

and the total disturbance in the form

$$\zeta(r, \theta) = \zeta_0 \sum_{m=0}^{\infty} \epsilon_m i^m \chi_m(r) \cos m\theta. \tag{1.2}$$

The cylindrical-wave representation of (1.1*b*) follows from the plane-wave representation of (1.1*a*) by virtue of Jacobi's expansion, wherein ϵ_m is Neumann's symbol:

$$\epsilon_0 = 1, \quad \epsilon_m = 2 \quad (m \geqslant 1). \tag{1.3}$$

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§ This study was motivated by the proposed design for an artificial island offshore from the Scripps Institution of Oceanography.

The wave number k is related to σ through the dispersion equation

$$k \tanh kd = \kappa \equiv \sigma^2/g. \tag{1.4}$$

We now introduce the displacement potential ϕ (the corresponding velocity potential is $-i\sigma\phi$) in the form

$$\phi(r,\theta,z) = \zeta_0 \sum_{m=0}^{\infty} \epsilon_m i^m \psi_m(r,z) \cos m\theta, \tag{1.5}$$

such that

$$\chi_m(r) = (\partial\psi_m/\partial z)_{z=d}. \tag{1.6}$$

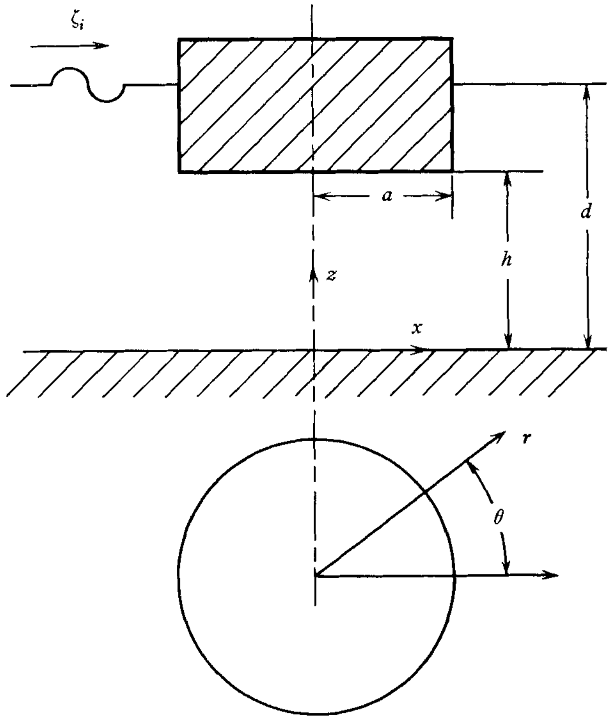


FIGURE 1. Circular dock of draft $d-h$ in water of depth d (supporting structure not considered).

Invoking the hypothesis of irrotational flow and the linearized free-surface condition and requiring the normal displacement to vanish at the rigid boundaries, we obtain the following boundary-value problem:

$$\nabla^2\phi = 0, \tag{1.7}$$

$$\phi_z = 0 \quad (z = 0), \tag{1.8}$$

$$\phi_z - \kappa\phi = 0 \quad (z = d, r \geqslant a), \tag{1.9}$$

$$\phi_r = 0 \quad (r = a, h < z \leqslant d), \tag{1.10}$$

and

$$\phi_z = 0 \quad (z = h, 0 \leqslant r < a). \tag{1.11}$$

We also require the scattered waves to satisfy the appropriate radiation and finiteness conditions as $r \to \infty$; in particular, ζ must have the asymptotic form

$$\zeta \sim \zeta_i + \zeta_0 (a/r)^{\frac{1}{2}} e^{ikr} A(\theta), \tag{1.12}$$

where $A(\theta)$ is the dimensionless *scattering amplitude* (cf. Morse & Feshbach 1953).

We seek especially the scattering amplitude, the differential scattering cross-section, $|A(\theta)|^2$, the total scattering cross-section,

$$Q = a\int_0^{2\pi} |A(\theta)|^2 d\theta, \tag{1.13}$$

the displacement amplitude on the periphery of the dock, $\zeta(a,\theta)$, and the lateral force on the dock, X.

We attack the boundary-value problem posed in (1.7)–(1.11) by constructing (in §2) representations of ϕ in the interior domain (under the island) and the exterior domain ($r > a$) in terms of the radial displacement, say $f(z,\theta)$, on the cylindrical interface $r = a$, $0 \leqslant z < h$, such that the solution reduces to that for a circular cylinder ($h = 0$) if $f(z,\theta) \equiv 0$. We then (in §3) construct a Schwinger-type variational approximation (Levine & Schwinger 1948) to the surface-wave amplitude that is: (*a*) stationary with respect to first-order variations of $f(z,\theta)$ about the true solution and (*b*) invariant under a scale transformation of $f(z,\theta)$. We then go on to calculate the scattering amplitude (in §4) and the peripheral displacement and the lateral force (in §5).

2. Modal expansions

We obtain separate representations of ψ_m in the interior domain, $0 \leqslant r < a$ and $0 \leqslant z \leqslant h$, and the exterior domain, $r \geqslant a$ and $0 \leqslant z \leqslant d$, in terms of the auxiliary function $f_m(z)$, such that

$$(\partial\psi_m/\partial r)_{r=a} = f_m(z) \quad (0 \leqslant z < h) \tag{2.1a}$$

$$= 0 \qquad (h < z \leqslant d). \tag{2.1b}$$

We then determine f_m from the requirement that ψ_m be continuous at $r = a$ in $0 \leqslant z \leqslant h$.

Solving Laplace's equation, (1.7), by separation of variables, we find that the most general solutions consistent with the representation (1.5) and the respective boundary conditions may be constructed by superposition of the modal solutions,

$$\phi_{mn} = I_m(n\pi r/h)\cos(n\pi z/h)\cos m\theta \quad (n = 1, 2, \ldots) \tag{2.2a}$$

$$= (r/a)^m \cos m\theta \quad (n = 0), \tag{2.2b}$$

which satisfy (1.8) and (1.11) and are bounded in $r = [0, a]$, and

$$\phi_m = K_m(\alpha r)\cos\alpha z\cos m\theta \quad (\alpha > 0) \tag{2.3a}$$

$$= \{J_m(kr), H_m(kr)\}\cosh kz\cos m\theta \quad (\alpha = -ik), \tag{2.3b}$$

which satisfy (1.8) and (1.9) if α is a root of

$$\alpha\tan\alpha d + \kappa = 0. \tag{2.4}$$†

† The roots of (2.4) are given by $\alpha = (p\pi/d) - (\kappa/p\pi)$, $p = 1, 2, \ldots$, with a maximum error of 1 % for $\kappa d = 1$ and of less than 1 % for all but the lowest ($p = 1$) mode for $\kappa d < 10$.

J_m is an ordinary Bessel function, and the corresponding ϕ_{mk} belong to the incident wave; H_m is a Hankel function of the first kind (we may omit the conventional superscript 1 without ambiguity), and the corresponding ϕ_{mk} represent outgoing surface waves, which decay exponentially with distance from the free surface; K_m is a modified Bessel function, and the corresponding $\phi_{m\alpha}$ represent trapped internal waves, which decay exponentially as $r \to \infty$.

We determine ψ_m in the interior domain by expanding $f_m(z)$ in the Fourier series

$$f_m(z) = \sum_{n=0}^{\infty} \epsilon_n F_{mn} \cos(n\pi z/h), \tag{2.5}$$

where
$$F_{mn} = (1/h)\int_0^h f_m(z)\cos(n\pi z/h), \quad F_{00} \equiv 0. \tag{2.6a, b}$$

The required expansion of ψ_m, as determined by (2.1 a), then is

$$\psi_m(r,z) = \frac{aF_{m0}}{m}\left(\frac{r}{a}\right)^m + 2\sum_{n=1}^{\infty}\frac{F_{mn}I_m(n\pi r/h)\cos(n\pi z/h)}{(n\pi/h)I'_m(n\pi a/h)} \quad (0 \leqslant r \leqslant a, \quad 0 \leqslant z \leqslant h). \tag{2.7}$$

We could add a constant to ψ_0, corresponding to the mode obtained by setting $m = 0$ in (2.2 b), but the resulting modification of the subsequent formulation would be trivial provided that the constraint implied by (2.6 b), namely

$$\int_0^h f_0(z)\,dz = 0, \tag{2.8}$$

is satisfied.

We find it expedient, in determining ψ_m in the exterior domain, to introduce the normalizing factors

$$N_\alpha = \tfrac{1}{2}[1 + (2\alpha d)^{-1}\sin 2\alpha d] \tag{2.9a}$$

and
$$N_k = \tfrac{1}{2}[1 + (2kd)^{-1}\sinh 2kd]; \tag{2.9b}$$

then, the functions
$$Z_\alpha(z) = N_\alpha^{-\frac{1}{2}}\cos\alpha z \tag{2.10a}$$

and
$$Z_k(z) = N_k^{-\frac{1}{2}}\cosh kz \tag{2.10b}$$

have mean-square values of unity and form a complete orthogonal set in $z = (0, d)$ if the spectrum of α is defined to include each of the infinite, discrete set of positive roots of (2.4) and, in addition, the negative imaginary root $\alpha = -ik$, for which (2.4) is equivalent to (1.4) and

$$K_m(-ikr) = \tfrac{1}{2}\pi i^{m+1} H_m(kr). \tag{2.11}$$

The solution for ψ_m in $r > a$ must be of the form

$$\psi_m = \psi_{mi} + \psi_{ms} \quad (r \geqslant a), \tag{2.12}$$

where
$$\psi_{mi}(r,z) = J_m(kr)[Z_k(z)/Z'_k(d)] \tag{2.13}$$

represents the incident wave [as may be shown by substituting (2.13) into (1.2) by way of (1.6)], and ψ_{ms} comprises only outgoing and trapped waves that must be determined by (2.1 a, b). Expanding the right-hand side of (2.1 a, b) in the $Z_\alpha(z)$, we find that the required expansion of ψ_m is given by

$$\psi_m(r,z) = \{J_m(kr) - [J'_m(ka)/H'_m(ka)]\,H_m(kr)\}\,[Z_k(z)/Z'_k(d)] + \sum_\alpha \mathscr{F}_{m\alpha}[\alpha K'_m(\alpha a)]^{-1} K_m(\alpha r)\, Z_\alpha(z), \quad (2.14)$$

where
$$\mathscr{F}_{m\alpha} = (1/d)\int_0^h f_m(z)\, Z_\alpha(z)\, dz, \quad (2.15)$$

and, here and subsequently, the summation over α comprises each of the positive roots of (2.4) and, in addition, $\alpha = -ik$ ($Z_\alpha \equiv Z_k$ and $\mathscr{F}_{m\alpha} \equiv \mathscr{F}_{mk}$ for $\alpha = -ik$).

Equating the representations of (2.7) and (2.14) at $r = a$, dividing the result through by a, and introducing

$$F_m = [aZ'_k(d)]^{-1}\{J_m(ka) - [J'_m(ka)/H'_m(ka)]\,H_m(ka)\} = 2i[\pi k a^2 H'_m(ka)\, Z'_k(d)]^{-1}, \quad (2.16)$$

$$G_m(x) = I_m(x)/xI'_m(x) \quad (2.17)$$

and
$$\mathscr{G}_m(x) = -K_m(x)/xK'_m(x), \quad (2.18)$$

we obtain

$$F_m Z_k(z) = \sum_\alpha \mathscr{F}_{m\alpha}\,\mathscr{G}_m(\alpha a)\, Z_\alpha(z) + m^{-1}F_{m0} + 2\sum_{n=1}^{\infty} F_{mn}\, G_m(n\pi a/h)\cos(n\pi z/h) \quad (0 \leqslant z \leqslant h) \quad (2.19)$$

as the determining equation for $f_m(z)$. Substituting the integral representations of $\mathscr{F}_{m\alpha}$ and F_{mn} from (2.6) and (2.15) into (2.19), we obtain the Fredholm integral equation

$$F_m Z_k(z) = \int_0^h g_m(z,\zeta) f_m(\zeta)\, d\zeta \quad (0 \leqslant z \leqslant h), \quad (2.20)$$

where

$$g_m(z,\zeta) = d^{-1}\sum_\alpha \mathscr{G}_m(\alpha a)\, Z_\alpha(z)\, Z_\alpha(\zeta) + h^{-1}\sum_{n=0}^{\infty} \epsilon_n G_m(n\pi a/h)\cos(n\pi z/h)\cos(n\pi\zeta/h) \quad (2.21)$$

is a symmetrical kernel. We remark that $G_m(0) = 1/m$ for $m \geqslant 1$ and that the term $n = 0$ must be omitted from the last summation if $m = 0$, in which case we also must invoke the constraint (2.8). We also notice that the asymptotic approximation (cf. Erdélyi *et al.* 1953, §7.13.2)

$$\mathscr{G}_m(\alpha a) \sim G_m(\alpha a) \sim (m^2 + \alpha^2 a^2)^{-\frac{1}{2}} \quad (2.22)$$

is exact at $\alpha a = 0$ for $m \geqslant 1$ and provides a fairly good interpolation for all real α, although not for $\alpha = -ika$ (unless $ka \gg m$).

3. Variational formulation

We now construct a variational approximation for the surface-wave amplitude in the re-normalized form

$$\lambda_m = \mathscr{F}_{mk}/F_m. \tag{3.1}$$

Multiplying both sides of (2.20) through by $f_m(z)$, integrating over $z = (0, h)$, dividing the result through by $d\mathscr{F}^2_{mk}$, and invoking the integral representation (2.15) for $\mathscr{F}_{mk}$ on the right-hand side, and (3.1) on the left-hand side, of the result, we obtain

$$\frac{1}{\lambda_m} = \frac{d\int_0^h\int_0^h f_m(z)\,g_m(z,\zeta)f_m(\zeta)\,d\zeta\,dz}{\left[\int_0^h f_m(z)\,Z_k(z)\,dz\right]^2}. \tag{3.2}$$

Invoking the standard variational procedure, we find that the right-hand side of (3.2) is stationary with respect to first-order variations of $f_m(z)$ about the true solution to (2.20). It also is invariant under a scale transformation of $f_m(z)$.

Now, let $f^*_m(z)$ be a trial function of arbitrary scale and let F^*_{mn} and $\mathscr{F}^*_{m\alpha}$ be the corresponding Fourier coefficients, as obtained by replacing f_m by f^*_m in (2.6) and (2.15). Substituting f^*_m and g_m from (2.21) into (3.2), we obtain

$$\lambda_m = \mathscr{F}^{*2}_{mk}\left[\sum_\alpha \mathscr{G}_m(\alpha a)\,\mathscr{F}^{*2}_{m\alpha} + 2(h/d)\sum_{n=1}^{\infty}\epsilon_n G_m(n\pi a/h)\,F^{*2}_{mn}\right]^{-1}. \tag{3.3}$$

Substituting $f_m(z) = Cf^*_m(z)$ into (3.1) to determine the scale constant C, we obtain

$$f_m(z) = (\lambda_m F_m/\mathscr{F}^*_{mn})\,f^*_m(z). \tag{3.4}$$

We could obtain $f_m(z)$ to any desired approximation by truncating the Fourier series (2.5) at $n = N$, substituting into (3.2), and requiring the result to be stationary with respect to first-order variations of each of $F_{m0}, F_{m1}, \ldots, F_{mN}$ to obtain $N+1$ linear equations in $F_{m0}, F_{m1}, \ldots, F_{mN}$ (N equations if $m = 0$, since $F_{00} \equiv 0$); however, this is tantamount to, but less direct than, the solution of (2.19) by Galerkin's method. We prefer to exploit the variational formulation more directly by introducing an assumed trial function into (3.3).

Remarking that the radial displacement $f_m(z)\cos m\theta$ is excited by the incident-wave component $\psi_m(z)$, which exhibits the z-dependence $Z_k(z)$, we choose $f^*_m = Z_k$ for $m \geqslant 1$ and add a constant term for $m = 0$ in order to satisfy (2.8):

$$f^*_m(z) = N_k^{-\frac{1}{2}}[\cosh kz - \delta_{m0}(kh)^{-1}\sinh kh], \tag{3.5}$$

where δ_{m0} is the Kronecker delta. Substituting (3.5) into (2.6) and (2.15), we obtain

$$F^*_{mn} = (-)^n N_k^{-\frac{1}{2}}(n^2\pi^2 + k^2h^2)^{-1}kh\sinh kh, \tag{3.6}$$

$$\mathscr{F}^*_{m\alpha} = (N_\alpha N_k)^{-\frac{1}{2}}\{(\alpha^2d^2 + k^2d^2)^{-1}[\alpha d\sin\alpha h\cosh kh + kd\cos\alpha h\sinh kh] - \delta_{m0}(\alpha dkh)^{-1}\sin\alpha h\sinh kh\}, \tag{3.7}$$

and

$$\mathscr{F}^*_{mk} = \tfrac{1}{2}(h/d)N_k^{-1}[1 + (2kh)^{-1}\sinh 2kh - 2\delta_{m0}(kh)^{-2}\sinh^2 kh]. \tag{3.8}$$

The variational approximation based on the trial function of (3.5) has the virtue of yielding the exact results for the scattering cross-section and the lateral force on the dock in the limit $ka \to \infty$. More accurate results for small values of ka might be obtained by using a trial function that reproduces the singularity in $f(z)$ at $z = h$, but this singularity is so weak that the differences are unlikely to be

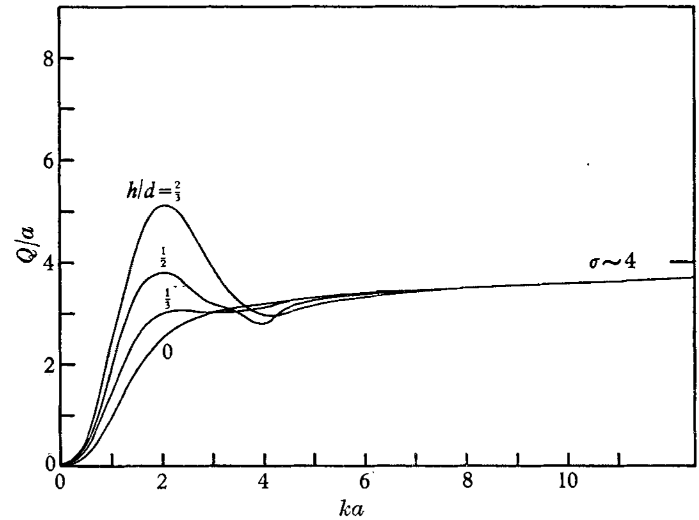


FIGURE 2*a*. The total scattering cross-section, as given by (4.4), for $d/a = \frac{3}{4}$. $\sigma = Q/a$.

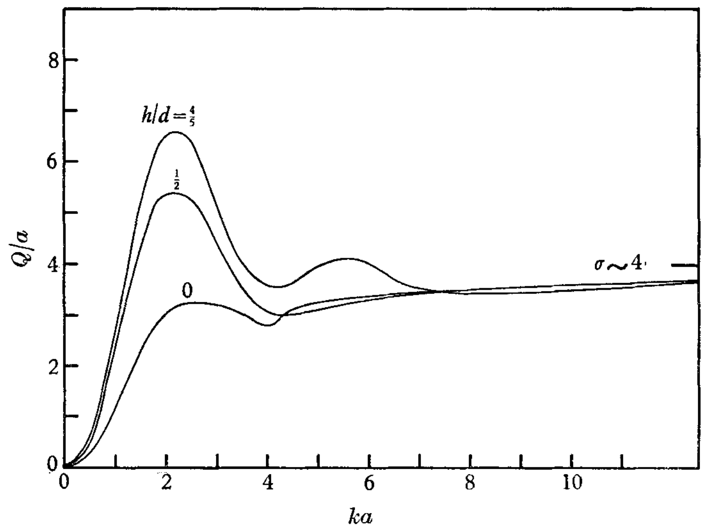


FIGURE 2*b*. The total scattering cross-section, as given by (4.4), for $d/a = \frac{1}{2}$. $\sigma = Q/a$.

important; and, in any event, the results for $ka \ll 1$ are of limited interest in the present context.

The numerical results given in §§4 and 5 below are based on the variational approximation to λ_m, as obtained by substituting (3.6)–(3.8) into (3.3).

4. Scattering amplitude

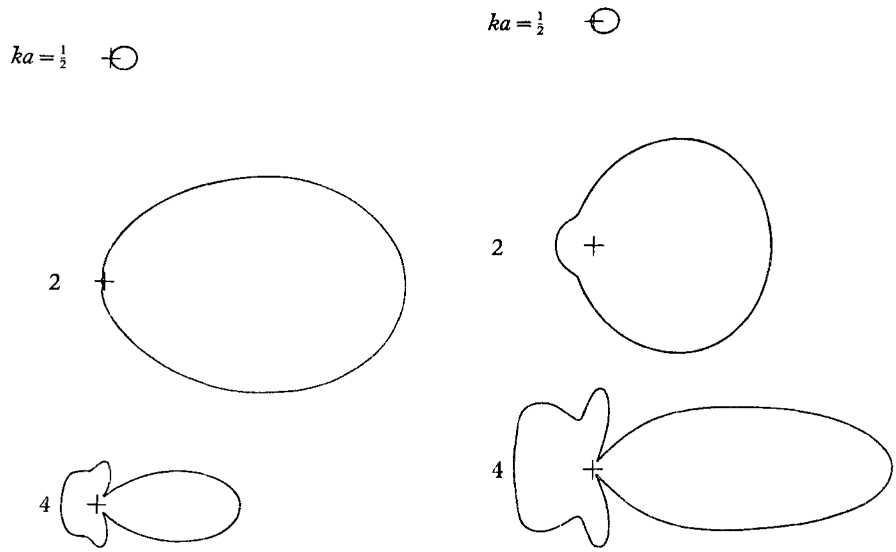
Substituting (2.14) into (1.6) and letting $kr \to \infty$, we obtain

$$\chi_m(r) \sim J_m(kr) + \{-[J'_m(ka)/H'_m(ka)] + [\mathscr{F}_{mk} Z'_k(d)/kH'_m(a)]\}(2/\pi kr)^{\frac{1}{2}} e^{i(kr-\frac{1}{2}m\pi-\frac{1}{4}\pi)} \quad (kr \to \infty). \tag{4.1}$$

Substituting $\mathscr{F}_{mk}$ into (4.1) from (3.1) and (2.16*b*), substituting the resulting expression for χ_m into (1.2), and comparing the resulting expression for ζ to (1.1*b*) and (1.12), we obtain the scattering amplitude in the form

$$A(\theta) = \sum_{m=0}^{\infty} \epsilon_m A_m \cos m\theta, \tag{4.2}$$

where $A_m = (2/\pi ka)^{\frac{1}{2}}\{-[J'_m(ka)/H'_m(ka)]e^{-i\pi/4} + (2\lambda_m/\pi)[kaH'_m(ka)]^{-2} e^{i\pi/4}\}$. (4.3)



(*a*) $h/d = \frac{2}{3}$ (*b*) $h/d = \frac{1}{3}$

FIGURE 3. The differential scattering cross-section, $|A(\theta)|^2$. The cross indicates the centre of the dock, and the incident wave is travelling from left to right.

Substituting (4.2) into (1.13), we obtain the total scattering cross-section in the form

$$Q = 2\pi a \sum_{0}^{\infty} \epsilon_m A_m^2. \tag{4.4}$$

We remark that the second term in (4.3) is asymptotically negligible compared with the first as $ka \to \infty$, in consequence of which the asymptotic value of Q is that for a circular cylinder, namely (cf. Morse & Feshbach 1953, p. 1381)

$$Q \sim 4a \quad (ka \to \infty). \tag{4.5}$$

We give graphical results for the total scattering cross-section, Q/a, in figures 2*a*, *b* and for the differential scattering cross-section, $|A(\theta)|^2$, in figures 3*a*, *b*. We call attention to the resonant peaks in both Q/a and $|A(\theta)|^2$ near $ka = 2$.

These peaks become more pronounced as the draft, $d-h$, decreases (see especially the results for $h/d = 4/5$ in figure 2*b*) and are characteristic of the limiting case of scattering by a rigid disk ($d-h \to 0$); on the other hand, they disappear in the limit $h \to 0$, wherein the scattering tends to that for a circular cylinder.

5. Peripheral disturbance

The ratio of the wave amplitude at the dock to that of the incident wave, obtained by setting $r = a$ in (1.2), is of special interest. Substituting (2.14) into (1.6) and introducing

$$\chi_m^{(0)}(a) = 2i[\pi ka H_m'(ka)]^{-1}, \tag{5.1}$$

we obtain

$$\chi_m(a) = \chi_m^{(0)}(a) - a \sum_\alpha \mathscr{F}_{m\alpha} \mathscr{G}_{m\alpha}(\alpha a) Z_\alpha'(d). \tag{5.2}$$

Substituting

$$a\mathscr{F}_{mk} = \lambda_m \chi_m^{(0)}(a)/Z_k'(d) \tag{5.3}$$

from (3.1) and (2.16*b*), and [cf. (3.4)]

$$\mathscr{F}_{m\alpha} = \mathscr{F}_{mk}(\mathscr{F}^*_{m\alpha}/\mathscr{F}^*_{mk}) \tag{5.4}$$

into (5.2), we obtain

$$\chi_m(a) = \chi_m^{(0)}(a)\{1 - \lambda_m \sum_\alpha (\mathscr{F}^*_{m\alpha}/\mathscr{F}^*_{mk})[Z_\alpha'(d)/Z_k'(d)]\mathscr{G}_m(\alpha a)\}. \tag{5.5}$$

We remark that $\chi_m^{(0)}(a)$ is the amplitude ratio in the special case $h = 0$.

The peripheral displacement obtained by substituting (5.5) into (1.2) is plotted in figures 4*a*, *b*. We observe that the variation of the displacement with ka in the illuminated and shadow zones follows classical behaviour.

The wave-induced force on the dock is in (or opposite to) the direction of the incident wave and is given by the real part of $X \exp(-i\sigma t)$, where

$$X = \rho\sigma^2\zeta_0 a \int_0^{2\pi} \int_h^d \phi(a, \theta, z) \cos\theta \, d\theta \, dz \tag{5.6a}$$

$$= 2\pi i \rho\sigma^2\zeta_0 a \int_h^d \psi_1(a, z) \, dz, \tag{5.6b}$$

and (5.6*b*) follows from (5.6*a*) by virtue of (1.5). Setting $r = a$ and $m = 1$ in (2.14) and invoking (5.1), (5.3) and (5.4), we obtain

$$\psi_1(a, z) = (k \sinh kd)^{-1} \chi_1^{(0)}(a)\{\cosh kz - \lambda_1 \sum_\alpha (\mathscr{F}^*_{1\alpha}/\mathscr{F}^*_{1k})(N_k/N_\alpha)^{\frac{1}{2}} \mathscr{G}_1(\alpha a) \cos \alpha z\}. \tag{5.7}$$

Substituting (5.7) into (5.6*b*) and dividing the result through by the displacement weight,

$$W = \pi a^2(d-h)\rho g, \tag{5.8}$$

we obtain

$$\frac{X}{W} = \frac{2i\kappa\zeta_0\chi_1^{(0)}(a)}{ka \sinh kd} \left\{ \frac{\sinh kd - \sinh kh}{kd - kh} - \lambda_1 \sum_\alpha \left(\frac{\mathscr{F}^*_{1\alpha}}{\mathscr{F}^*_{1k}}\right) \left(\frac{N_k}{N_\alpha}\right)^{\frac{1}{2}} \mathscr{G}_1(\alpha a) \left(\frac{\sin \alpha d - \sin \alpha h}{\alpha d - \alpha h}\right) \right\}. \tag{5.9}$$

The result (5.9) is plotted in figures 5*a*, *b*. We observe that the maximum value of X/W increases with increasing draft, as also does the wavelength at which this maximum occurs. A practical design criterion appears to involve a compromise. As the draft of the dock increases the scattering cross-section, Q, decreases, but the wave-induced force, X/W, increases. Small Q is desirable for near-shore docks in order to minimize changes in beach-transport processes; on the other

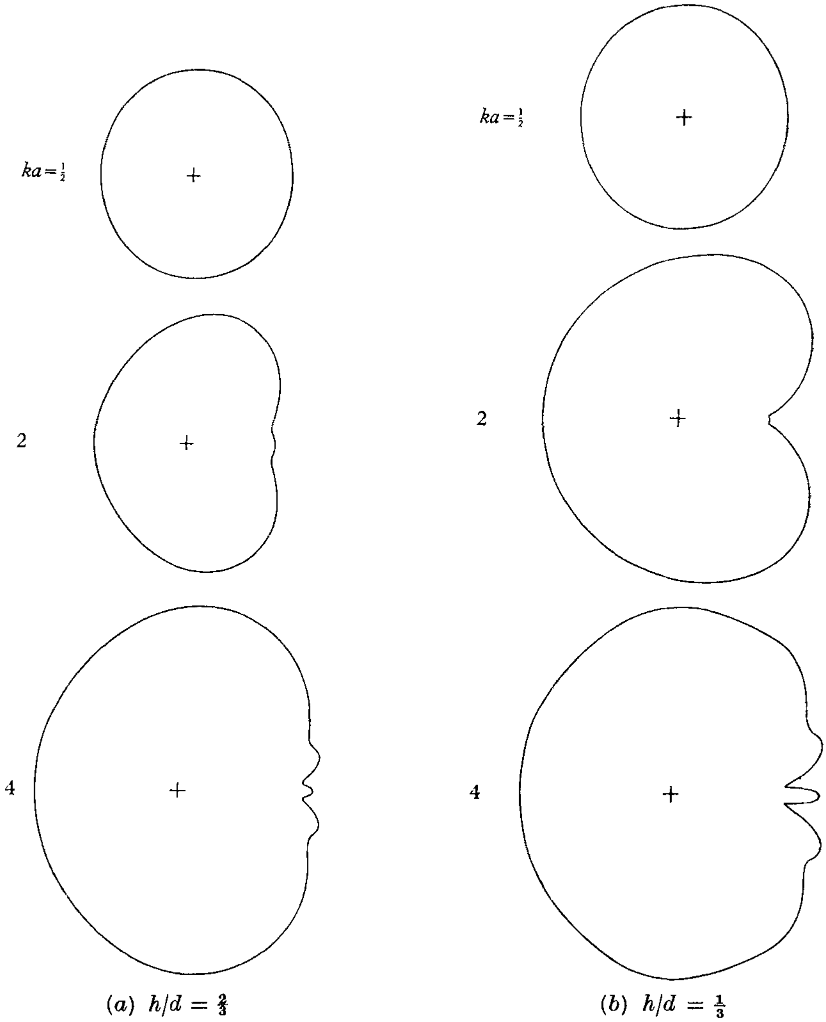


FIGURE 4. The peripheral displacement relative to that of the incident wave. The cross indicates the centre of the dock, and the incident wave is travelling from left to right.

hand, small X/W is desirable so that design safety factors can be optimized. The theoretical and numerical results presented here should be a useful engineering aid in reaching a practical compromise.

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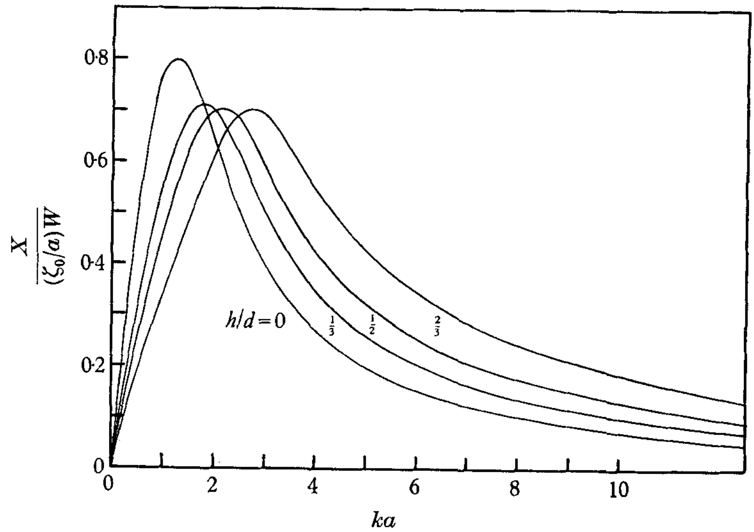

FIGURE 5*a*. The lateral force on the dock relative to the product of the buoyancy force and the amplitude ratio ζ_0/a, as given by (5·9), for $d/a = \frac{3}{4}$.

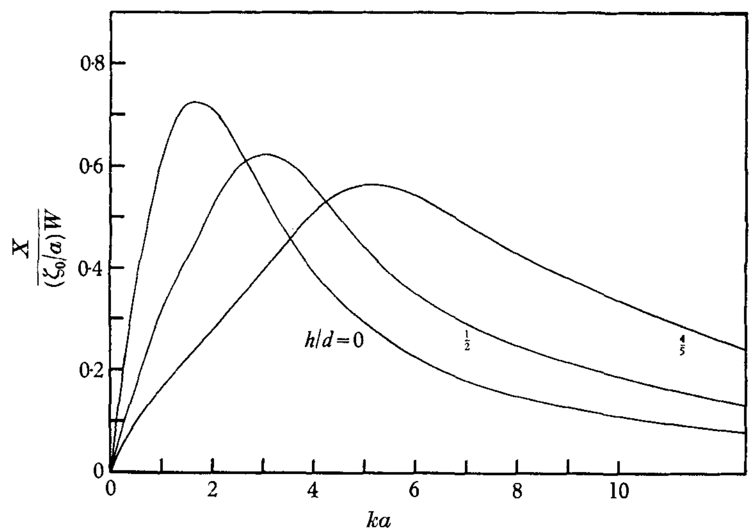

FIGURE 5*b*. The lateral force on the dock relative to the product of the buoyancy force and the amplitude ratio ζ_0/a, as given by (5·9), for $d/a = \frac{1}{2}$.

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